ALUPE UNIVERSITY

BASTION OF KNOWLEDGE

OFFICE OF THE DEPUTY VICE CHANCELLOR
ACADEMICS, RESEARCH AND STUDENT AFFAIRS

UNIVERSITY EXAMINATIONS

2025/2026 ACADEMIC YEAR

THIRD YEAR FIRST SEMESTER EXAMINATION

FOR THE DEGREE OF BACHELOR OF ARTS IN EDUCATION

MAIN EXAM

COURSE CODE: MAT 311

COURSE TITLE: LINEAR ALGEBRA II

DATE: 15/12/2025 TIME: 2:00PM-5:00PM

INSTRUCTION TO CANDIDATES

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MAIN EXAM

MAT 311: LINEAR ALGEBRA II

STREAM: BACHELOR OF ARTS IN EDUCATION

DURATION: 3 Hours

INSTRUCTIONS TO CANDIDATES

i. *Answer **ALL** questions from section A and any **THREE** from section B.*

ii. *Maps and diagrams should be used whenever they serve to illustrate the answer.*

iii. *Do not write on the question paper.*

SECTION A (31 MARKS) COMPULSORY

QUESTION ONE (16 MARKS)

a) Briefly define or explain the following concepts.
   (i) The minimal polynomial of a linear operator. **(3 marks)**
   (ii) The Cayley-Hamilton Theorem. **(2 marks)**

b) Let $f: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined as
$f([u_1, u_2, u_3, u_4]) = [u_1 + u_2, \ u_3 + u_4, \ u_1 + u_3]$
Find
   (i) Basis for range of f **(4 marks)**
   (ii) Basis for kernel of f **(4 marks)**
   (iii) Rank of (f) and nullity of (f) **(3 marks)**

QUESTION TWO (15 MARKS)

a) Find the eigenvalues of the matrix $B = \begin{pmatrix} 7 & 1 \\ 9 & 2 \end{pmatrix}$. **(4 marks)**

b) Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors:

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

   (i) Find an orthogonal basis for W using the Gram-Schmidt process. **(6 marks)**

   (ii) Find the distance from the vector $v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ to the subspace W. **(5 marks)**

SECTION B (39 MARKS)

QUESTION THREE (13 MARKS)

a) Solve the system of linear equations using Gaussian elimination:

$$x + 2y + 3z = 6$$
$$2x + 4y + z = 7$$
$$3x + y + 2z = 8$$

(5 marks)

b) Define the linear transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ by $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \left(\begin{bmatrix} a+b \\ b-c \\ a+d \end{bmatrix}\right)$

i. Find a basis for the null space of T and its dimension (3 marks)

ii. Describe the Range of T (3 marks)

iii. Find a basis for the range of T and its dimension (2 marks)

QUESTION FOUR (13 MARKS)

a) Let $C = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

(i) Find all eigenvalues of C. (2 marks)

(ii) For each eigenvalue, find a basis for the corresponding eigenspace. (4 marks)

(iii)Determine if C is diagonalizable. If yes, find an invertible matrix P and a diagonal matrix D such that $C = P\,DP^{-1}$ (4 marks)

b) Let $V = \mathbb{R}^3$. Show that W is a subspace of V where:
$W = \{(a, b, 0): a, b \in \mathbb{R}\}$, that is W is the x, y plane consisting of the vectors whose third component is zero. (3 marks)

QUESTION FIVE (13 MARKS)

a) Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

(6 marks)

b) Is the matrix A from part (a) diagonalizable? Justify your answer. If yes, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. (4 marks)

c) Compute A^5 using the diagonalization from part (b). (You do not need to simplify the numerical expressions). **(3 marks)**

QUESTION SIX **(13 MARKS)**

Let $\mathbb{R}^3$ be equipped with the standard dot product. Consider the subspace

$$W = \{(x, y, z) \in \mathbb{R}^3 : x + 2y - z = 0\}$$

a) Find a basis for W. **(3 marks)**
b) Use the Gram-Schmidt process on the basis you found in (a) to obtain an orthonormal basis for W. **(5 marks)**
c) Find the orthogonal projection of the vector $v = (1, 1, 1)$ onto the subspace W.

(3 marks)

d) Find the distance from v to W. **(2 marks)**

QUESTION SEVEN **(13 MARKS)**

a) Consider the quadratic form

$$Q(x_1, x_2, x_3) = 2{x_1}^2 + 3{x_2}^2 + 2{x_3}^2 - 2x_1x_2 + 4x_1x_3 - 2x_2x_3$$

(i) Write the quadratic form in the form $x^T Ax$, where A is a symmetric matrix. **(3 marks)**
(ii) Find an orthogonal matrix P such that $P^T AP$ is a diagonal matrix. **(4 marks)**
(iii) Classify the quadratic form (positive definite, negative definite, indefinite, etc.).
(2 marks)

b) Consider the matrix $A = \begin{pmatrix} 7 & 3 \\ 1 & 2 \end{pmatrix}$.

(i) Find the characteristic polynomial $p(\lambda)$ of the matrix A. **(2 marks)**

(ii) Verify the Cayley-Hamilton Theorem for A. **(2 marks)**

________________END OF EXAM________________