ALUPE UNIVERSITY

OFFICE OF THE DEPUTY VICE CHANCELLOR

ACADEMICS, RESEARCH AND STUDENTS AFFAIRS

UNIVERSITY EXAMINATIONS

2022/2023 ACADEMIC YEAR

FIRST YEAR FIRST SEMESTER REGULAR MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF EDUCATION ARTS/SCIENCE

COURSE CODE: **MAT 311E**

COURSE TITLE: **REAL ANALYSIS II**

DATE: **19TH DEC 2022** **TIME:** **2.00PM – 5.00PM**

INSTRUCTION TO CANDIDATES

- **SEE INSIDE**

THIS PAPER CONSISTS OF 4 PRINTED PAGES **PLEASE TURN OVER**

MAT 310/311E: REAL ANALYSIS II

STREAM: BED (Arts/Science) **DURATION: 3 Hours**

INSTRUCTIONS TO CANDIDATES

i. *Answer ALL Questions from* ***section A*** *and any* ***THREE*** *from* ***section B****.*

ii. *Do not write on the question paper.*

SECTION A (31 Marks)

Answer ALL questions from this section.

Question One (16 Marks)

a) Show that an arbitrary union of open sets is open, and a finite intersection of open sets is open. (5 Marks)

b) Let $f : A \rightarrow R$ and $g : B \rightarrow R$ where $f(A) \subset B$. Prove that if f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f : A \rightarrow R$ is continuous at c. (4 Marks)

c) Suppose that $f_n: A \rightarrow R$ is bounded on A for every $n \in N$ and $f_n \rightarrow f$ uniformly on A. Then $f : A \rightarrow R$ is bounded on A. (4 Marks)

d) Show that the constant function $f(x) = 1$ on $[0, 1]$ is Riemann integrable, and $\int_0^1 1\, dx = 1$. (3 Marks)

Question Two (15 Marks)

a) Show that if $f : A \subset R \rightarrow R$ has a local extreme value at an interior point $c \in A$ and f is differentiable at c, then $f'(c) = 0$. (4 Marks)

b) Find the derivative of the function $f : R \rightarrow R$ defined by $f(x) = \begin{cases} 1/x & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$. (3 Marks)

c) Prove that the function $f(x) = x^2$ is continuous but not uniformly continuous on R. (4 Marks)

d) Prove that the limit of a function is unique if it exists. (4 Marks)

SECTION B (39 Marks)

Answer any THREE questions from this section.

Question Three (13 Marks)

a) Let $A = [0, \infty) \setminus \{9\}$ and define $f : A \to R$ by $f(x) = \frac{x-9}{\sqrt{x}-3}$. Show that $\lim_{x \to 9} f(x) = 6$. (4 Marks)

b) Define the following terms

i) Open set

ii) Connected sets

iii) Continuous function (6 Marks)

c) Show that the function $f : [0, \infty) \to R$ defined by $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$. (3 Marks)

Question Four (13 Marks)

a) Prove that a set of real numbers is connected if and only if it is an interval.

(7 Marks)

b) A function $f : A \to R$ is not uniformly continuous on A if and only if there exists $\epsilon_0 > 0$ and sequences $(x_n), (y_n)$ in A such that $\lim_{n \to \infty} |xn - yn| = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n \in N$. (6 Marks)

Question Five (13 Marks)

a) Suppose that $f : [a, b] \to R$ is a continuous function on a closed, bounded interval. Prove that $f([a, b]) = [m, M]$ is a closed, bounded interval. (4 Marks)

b) Show that the function $f : R \to R$ defined by $f(x) = x^{1/3}$ is differentiable at $x \neq 0$ with $f'(x) = \frac{1}{3x^{2/3}}$. (5 Marks)

c) Suppose that $f : [a, b] \to R$ is continuous on the closed, bounded interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$. Prove that there exists $a < c < b$ such that $f'(c) = 0$. (4 Marks)

Question Six (13 Marks)

a) Prove that a sequence (f_n) of functions $f_n : A \to R$ converges uniformly on A if and only if it is uniformly Cauchy on A. (9 marks)

b) Suppose that $f : [a, b] \to R$ is continuous on the closed, bounded interval $[a, b]$ and differentiable on the open interval (a, b). Prove that there exists $a < c < b$ such that $f'c) = \frac{f(b) - f(a)}{b - a}$. (4 Marks)

Question Seven (13 Marks)

a) Prove that a monotonic function $f : [a, b] \rightarrow R$ on a compact interval is Riemann integrable. (8 Marks)

b) Suppose that $f, g : [a, b] \rightarrow R$ are integrable and $f \leq g$. Prove that $\int_a^b f \leq \int_a^b g$. (5 Marks)

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